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Anderson's inequality on time scales

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Abstract

We establish Anderson's inequality on time scales as follows:

$$\int_0^1 \left(\prod_{i=1}^n f_i^\sigma(t)\right)\Delta t \geq \left(\int_0^1 (t+\sigma(t))^n \Delta t\right)\left(\prod_{i=1}^n \int_0^1 f_i(t)\Delta t\right) \geq \left(2^n \int_0^1 t^n \Delta t\right)\left(\prod_{i=1}^n \int_0^1 f_i(t)\Delta t\right)$$

if f_i $(i = 1, \ldots, n)$ satisfy some suitable conditions.

Keywords: Time scales; Anderson's inequality; Delta differentiable

1. Introduction

In 1958, Anderson [1] showed the following very interesting inequality:

$$\int_0^1 F_1(x)F_2(x)\cdots F_n(x)\mathrm{d}x \geq \frac{2n}{n+1}\left(\int_0^1 F_1(x)\mathrm{d}x\right)\cdots\left(\int_0^1 F_n(x)\mathrm{d}x\right) \qquad (R_1)$$

if F_i is convex increasing on $[0, 1]$ and $F_i(0) = 0$ for each $i = 1, 2, \ldots, n$. Recently, Fink [2] improved Anderson's inequality (R_1) to the following form:

$$\int_0^1 F_1(x)F_2(x)\cdots F_n(x)\mathrm{d}x \geq \frac{2^n}{n+1}\left(\int_0^1 F_1(x)\mathrm{d}x\right)\cdots\left(\int_0^1 F_n(x)\mathrm{d}x\right) \qquad (R_2)$$

if $\frac{F_i(t)}{t}$ is increasing on $(0, 1]$ and $F_i(0) = 0$ for each $i = 1, 2, \ldots, n$. Moreover, Fink [2] also pointed out that the condition $F_i(0) = 0$ $(i = 1, 2, \ldots, n)$ cannot be dropped. The purpose of this note is to establish Anderson's inequality on time scales. For other related results, we refer the reader to the book [3] by Mitrinvić et al. Now, we

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briefly introduce the time scale theory and refer the reader to Aulbach and Hilger [4], Hilger [5] and the books [6] and [7] for further details.

Definition 1.A. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Throughout this work we assume that $\mathbb{T}$ is a time scale and $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\} \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\} \in \mathbb{T}.$$

If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$, we say t is left scattered. If $\sigma(t) = t$, we say t is right dense, while if $\rho(t) = t$, we say t is left dense.

Definition 1.B. For $a, b \in \mathbb{R}$ with $a \le b$, we define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : \ a \le t \le b\}.$$

Other types of interval are defined similarly.

Definition 1.C. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$; then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $x^{\Delta}(t)$ the **delta derivative** of $x(t)$ at $t \in \mathbb{T}$ and $x(t)$ delta differentiable at t. If $x(t)$ is delta differentiable at every point of $\mathbb{T}$, then we say $x(t)$ is delta differentiable at $\mathbb{T}$.

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Definition 1.D. A function $F : \mathbb{T}^{\kappa} \to \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. In this case, we define the integral of f by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s)$$

for $s, t \in \mathbb{T}^{\kappa}$, where

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} - \{m\}, \text{ if } \mathbb{T} \text{ has a left-scattered maximal point } m, \\ \mathbb{T}, \text{ otherwise.} \end{cases}$$

Definition 1.E. If $f : \mathbb{T} \to \mathbb{R}$, then $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^{\sigma}(t) = f(\sigma(t))$$

for all $t \in \mathbb{T}$.

Definition 1.F. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it satisfies the following conditions:

(A) f is continuous at each right-dense or maximal element $t \in \mathbb{T}$;
(B) the left-sided limit $\lim_{s \to t^-} f(s) = f(t^-)$ exists at each left-dense point $t \in \mathbb{T}$.

In this work, we write

$$C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}) = \{f \mid f : \mathbb{T} \to \mathbb{R} \text{ is a rd-continuous function}\}.$$

2. Main result

Lemma 2.1. *Let* $f, g \in C_{\mathrm{rd}}([0,1], \mathbb{R})$ *satisfy* $f(0) = 0$ *and* g *be increasing on* $(0, 1]$. *If* $\frac{f(t)}{t+\sigma(t)}$ *is increasing on* $(0, 1]$, *then*

$$\int_0^1 f(t)g^\sigma(t)\Delta t \geq \int_0^1 f^*(t)g^\sigma(t)\Delta t,$$

where $f^*(t) := (t + \sigma(t)) \int_0^1 f(u)\Delta u$ *for* $t \in [0, 1]$.

Proof. We define

$$H(t) := \int_0^t [f^*(x) - f(x)]\Delta x \quad \text{on } [0,1].$$

Then $H(0) = 0$ and

$$H(1) = \int_0^1 [(x + \sigma(x))\Delta x - 1] \int_0^1 f(u)\Delta u = [(x^2 |_0^1) - 1] \int_0^1 f(u)\Delta u = 0.$$

Moreover,

$$H^\Delta(0) = f^*(0) - f(0) = f^*(0) = \sigma(0) \int_0^1 f(u)\Delta u \geq 0$$

and

$$\begin{aligned} H^\Delta(t) &= f^*(t) - f(t) \\ &= (t + \sigma(t)) \int_0^1 f(u)\Delta u - f(t) \\ &= (t + \sigma(t)) \left\{ \int_0^1 f(u)\Delta u - \frac{f(t)}{t + \sigma(t)} \right\} \text{ on } (0, 1]. \end{aligned}$$

Since $\frac{f(t)}{t+\sigma(t)}$ is increasing on $(0, 1]$, we get that $\frac{H^\Delta(t)}{t+\sigma(t)}$ is decreasing on $(0, 1]$.

Next, we show that $H(t) \geq 0$ on $[0, 1]$. This proof can be divided into two parts.

(a) If there exists a $t_0 \in (0, 1)$ such that $\frac{H^\Delta(t_0)}{t_0+\sigma(t_0)} = 0$, then

$$H^\Delta(t) \geq 0 \quad \text{on } [0, t_0] \quad \text{and} \quad H^\Delta(t) \leq 0 \quad \text{on } [t_0, 1].$$

Hence, if $t \in [0, t_0]$, then $H(t) \geq H(0) = 0$. If $t \in [t_0, 1]$, then $H(t) \geq H(1) = 0$ on $[t_0, 1]$. Thus $H(t) \geq 0$ on $[0, 1]$.

(b) $\frac{H^\Delta(t)}{t+\sigma(t)} > 0$ on $(0, 1)$. In this case, $H(t)$ is increasing on $[0, 1)$. Hence, $H(t) \geq H(0) = 0$ on $[0, 1)$. It follows from $H(1) = 0$ that $H(t) \geq 0$ on $[0, 1]$. Thus, by Theorem 1.77 of [6],

$$\begin{aligned} \int_0^1 [f(t) - f^*(t)]g^\sigma(t)\Delta t &= -\int_0^1 H^\Delta(t)g^\sigma(t)\Delta t \\ &= -\left\{ H(t)g(t) |_0^1 - \int_0^1 g^\Delta(t)H(t)\Delta t \right\} \\ &= \int_0^1 g^\Delta(t)H(t)\Delta t \\ &\geq 0, \end{aligned}$$

because $H(t) \geq 0$ and g is increasing (and hence $g^\Delta(t) \geq 0$). Thus, we complete our proof. □

We are in a position to state and prove our main result as follows:

Theorem 2.2. *Suppose that* $t \in [0, 1]$ *does not satisfy the following two conditions:*

(1^0) *t is left dense and right scattered,*

(2^0) *t is right dense and left scattered.*

Let $f_1, f_2, \ldots, f_n \in C_{\mathrm{rd}}([0,1], \mathbb{R})$ *with* $f_i(0) = 0$ *and* $\frac{f_i(t)}{t+\sigma(t)}$ *increasing on* $(0,1]$, *for* $i = 1, 2, \ldots, n$*; then*

$$\begin{aligned}\int_0^1 \left(\prod_{i=1}^n f_i^\sigma(t)\right) \Delta t &\geq \left(\int_0^1 (t+\sigma(t))^n \Delta t\right) \left(\prod_{i=1}^n \int_0^1 f_i(t) \Delta t\right) \\ &\geq \left(2^n \int_0^1 t^n \Delta t\right) \left(\prod_{i=1}^n \int_0^1 f_i(t) \Delta t\right). \qquad (R_3)\end{aligned}$$

Proof. It follows from Lemma 2.1, $\sigma(t) \geq t$ and the increasing property of f_i, $i = 1, 2, \ldots, n$, that, for f_n^* defined as in Lemma 2.1,

$$\begin{aligned}\int_0^1 \left(\prod_{i=1}^n f_i^\sigma(t)\right) \Delta t &\geq \int_0^1 f_n(t) \left(\prod_{i=1}^{n-1} f_i^\sigma(t)\right) \Delta t \\ &\geq \int_0^1 f_n^*(t) \left(\prod_{i=1}^{n-1} f_i^\sigma(t)\right) \Delta t \\ &= \left(\int_0^1 f_n(t)\Delta t\right) \left(\int_0^1 (t+\sigma(t)) \left(\prod_{i=1}^{n-1} f_i^\sigma(t)\right) \Delta t\right) \\ &= \left(\int_0^1 f_n(t)\Delta t\right) \int_0^1 (\rho(t)+t)^\sigma \left(\prod_{i=1}^{n-1} f_i^\sigma(t)\right) \Delta t \quad (\text{using } \sigma(\rho(t)) = t) \\ &\geq \left(\int_0^1 f_n(t)\Delta t\right) \int_0^1 f_{n-1}(t) \left\{(\rho(t)+t)^\sigma \prod_{i=1}^{n-2} f_i^\sigma(t)\right\} \Delta t \\ &\geq \left(\int_0^1 f_n(t)\Delta t\right) \int_0^1 f_{n-1}^*(t) \left\{(\rho(t)+t)^\sigma \prod_{i=1}^{n-2} f_i^\sigma(t)\right\} \Delta t \\ &\geq \left(\int_0^1 f_n(t)\Delta t\right) \left(\int_0^1 f_{n-1}(t)\Delta t\right) \int_0^1 (t+\sigma(t)) \left\{(\rho(t)+t)^\sigma \prod_{i=1}^{n-2} f_i^\sigma(t)\right\} \Delta t \\ &= \left(\int_0^1 f_n(t)\Delta t\right) \left(\int_0^1 f_{n-1}(t)\Delta t\right) \int_0^1 \left\{[(\rho(t)+t)^\sigma]^2 \prod_{i=1}^{n-2} f_i^\sigma(t)\right\} \Delta t \\ &\geq \left(\int_0^1 f_n(t)\Delta t\right) \left(\int_0^1 f_{n-1}(t)\Delta t\right) \int_0^1 f_{n-2}(t) \left\{[(\rho(t)+t)^\sigma]^2 \prod_{i=1}^{n-3} f_i^\sigma(t)\right\} \Delta t \\ &\geq \cdots \\ &= \left(\prod_{i=1}^n \int_0^1 f_i(t)\Delta t\right) \left(\int_0^1 [(\rho(t)+t)^\sigma]^n \Delta t\right) \\ &= \left(\prod_{i=1}^n \int_0^1 f_i(t)\Delta t\right) \left(\int_0^1 (t+\sigma(t))^n \Delta t\right) \\ &\geq \left(2^n \int_0^1 t^n \Delta t\right) \left(\prod_{i=1}^n \int_0^1 f_i(t)\Delta t\right).\end{aligned}$$

Thus, we complete the proof of Theorem 2.2. □

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